The least positive integer N.

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Determine the least positive integer N such that for all integers $n > N$,.

$$(n^{n+1}/(n+1)^n)^n < n! < (n^{n+1}/(n+1)^n)^{n+1}$$

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Let $a_n := (n^{n+1}/(n+1)^n)^n, b_n := n!$ and $c_n := (n^{n+1}/(n+1)^n)^{n+1}, n \in \mathbb{N}$.

1. Proof of inequality $a_n < b_n, \forall n \in \mathbb{N}$ (by Math Induction).

First we will prove that for any $n \in \mathbb{N}$ holds inequality

(**1**) $\quad \dfrac{a_{n+1}}{a_n} < \dfrac{b_{n+1}}{b_n}$.

We have $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^{(n+2)(n+1)}}{(n+2)^{(n+1)^2}} \cdot \dfrac{(n+1)^{n^2}}{n^{(n+1)n}} = \dfrac{(n+1)^{n+2}(n^2+2n+1)^{n(n+1)}}{(n+2)^{n+1}(n^2+2n)^{n(n+1)}}$,

$\dfrac{b_{n+1}}{b_n} = n+1$ and, therefore, (**1**) $\Leftrightarrow \dfrac{(n+1)^{n+2}(n^2+2n+1)^{n(n+1)}}{(n+2)^{n+1}(n^2+2n)^{n(n+1)}} < n+1 \Leftrightarrow$

$\left(\dfrac{n^2+2n+1}{n^2+2n}\right)^{n(n+1)} < \left(\dfrac{n+2}{n+1}\right)^{n+1} \Leftrightarrow \left(\dfrac{n^2+2n+1}{n^2+2n}\right)^n < \dfrac{n+2}{n+1} \Leftrightarrow$

$1 - \dfrac{1}{n+2} < \left(1 - \dfrac{1}{n^2+2n+1}\right)^n$,

where latter inequality holds for any $n \in \mathbb{N}$ because by Bernoulli's Inequality

$\left(1 - \dfrac{1}{n^2+2n+1}\right)^n \geq 1 - \dfrac{n}{n^2+2n+1}$ and $-\dfrac{n}{n^2+2n+1} > -\dfrac{1}{n+2} \Leftrightarrow n(n+2) < n^2+2n+$

Since $a_1 = \dfrac{1}{2} < 1 = b_1$ and for any $n \in \mathbb{N}$ assuming $a_n < b_n$ and using inequality (**1**)

we obtain $a_{n+1} = a_n \cdot \dfrac{a_{n+1}}{a_n} < b_n \cdot \dfrac{b_{n+1}}{b_n} = b_{n+1}$.

Thus, by Math Induction $(n^{n+1}/(n+1)^n)^n < n!, \forall n \in \mathbb{N}$.

2. Proof of inequality $b_n < c_n, \forall n \in \mathbb{N} \setminus \{1\}$.

First we will prove that for any $n \in \mathbb{N}$ holds inequality

(**2**) $\quad \dfrac{b_{n+1}}{b_n} < \dfrac{c_{n+1}}{c_n}$.

Since $\dfrac{c_{n+1}}{c_n} = \dfrac{(n+1)^{(n+2)^2}}{(n+2)^{(n+1)(n+2)}} \cdot \dfrac{(n+1)^{n(n+1)}}{n^{(n+1)^2}} = \dfrac{(n+1)^{2(n+1)^2} \cdot (n+1)^{n+2}}{(n^2+2n)^{(n+1)^2}(n+2)^{n+1}}$ then

$\dfrac{b_{n+1}}{b_n} < \dfrac{c_{n+1}}{c_n} \Leftrightarrow n+1 < \dfrac{(n+1)^{2(n+1)^2} \cdot (n+1)^{n+2}}{(n^2+2n)^{(n+1)^2}(n+2)^{n+1}} \Leftrightarrow 1 < \dfrac{(n+1)^{2(n+1)^2} \cdot (n+1)^{n+1}}{(n^2+2n)^{(n+1)^2}(n+2)^{n+1}} \Leftrightarrow$

$1 < \dfrac{(n+1)^{2(n+1)} \cdot (n+1)}{(n^2+2n)^{n+1}(n+2)} \Leftrightarrow 1 + \dfrac{1}{n+1} < \left(1 + \dfrac{1}{n^2+2n}\right)^{n+1}$

where latter inequality holds for any $n \in \mathbb{N}$ because by Bernoulli's Inequality

$\left(1 + \dfrac{1}{n^2+2n}\right)^{n+1} > 1 + \dfrac{n+1}{n^2+2n}$ and $\dfrac{n+1}{n^2+2n} > \dfrac{1}{n+1} \Leftrightarrow 1 > 0.$

Since $n! < \left(\dfrac{n+1}{e}\right)^{n+1} e$ and $\left(\dfrac{n+1}{e}\right)^{n+1} e < (n^{n+1}/(n+1)^n)^{n+1} \Leftrightarrow$

$\dfrac{n+1}{e} e^{1/(n+1)} < n^{n+1}/(n+1)^n \Leftrightarrow \left(1 + \dfrac{1}{n}\right)^{n+1} < e^{1-1/(n+1)}$

$\dfrac{n+1}{e} e^{1/(n+1)} < n^{n+1}/(n+1)^n \Leftrightarrow \left(1 + \dfrac{1}{n}\right)^n$.

By direct calculation we can see that $b_2 = 2, c_2 = (2^3/3^2)^3 = \dfrac{512}{729} < 2 = b_2$,
$b_3 = 6, c_3 = (3^4/4^3)^4 = 2.5658 < 6 = b_3, \ldots, c_{16} < b_{16}$ ($c_{16} - b_{16} \approx -5.726 \times 10^{11}$).
But $b_{17} < c_{17}$ because $\left(17^{17+1}/(17+1)^{17}\right)^{17+1} - 17! \approx 8.1621 \times 10^{11}$.
Inequality $b_{17} < c_{17}$ can be taken as the *base of Math Induction*.
Step of Math Induction:
For any natural $n \geq 17$ assuming $b_n < c_n$ and using inequality (**2**)
we obtain $b_{n+1} = b_n \cdot \dfrac{b_{n+1}}{b_n} < c_n \cdot \dfrac{c_{n+1}}{c_n} = c_{n+1}$.
Thus, 16 is the least natural N such that for all natural $n > N$ holds inequality $n! < (n^{n+1}/(n+1)^n)^{n+1}$ and inequality $(n^{n+1}/(n+1)^n)^n < n!$ holds for any natural n.